

## RELATIVE IMAGINARY QUADRATIC FIELDS OF CLASS NUMBER 1 OR 2

BY

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**Abstract.** Let  $K$  be a normal totally real algebraic number field. It is shown how to effectively classify all totally imaginary quadratic extensions of class number 1. Let  $K$  be a real quadratic field of class number 1, whose fundamental unit has norm  $-1$ . Then it is shown how to effectively classify all totally imaginary quadratic extensions of class number 2.

**1. Introduction.** Let  $K$  be a totally real field, and let  $L$  be a totally imaginary quadratic extension of  $K$ . Holding  $K$  fixed, there are only a finite number of fields  $L$  having a given class number  $h_L$  [3]. For  $K = \mathbb{Q}$ , the problem of effectively determining all  $L$  having a given class number has its origins in Gauss' *Disquisitiones*. Although extensive work has been done over a period of many years, the first notable successes have come only fairly recently. In 1966, Stark [6] settled the problem  $K = \mathbb{Q}$ ,  $h_L = 1$ . In 1970, the author [4] reduced the problem  $K = \mathbb{Q}$ ,  $h_L = 2$  to a conjecture on linear forms in the logarithms of algebraic numbers. A weak form of this conjecture was proven by Baker [1] and Stark [6]. Their form of the conjecture sufficed to settle the class number 2 problem. In the present work we will consider the problems  $h_L = 1$  and  $h_L = 2$  for algebraic number fields other than  $\mathbb{Q}$ . The basic idea is to reduce the analytic difficulties back to the case  $K = \mathbb{Q}$ , where the situation has been carefully studied. Therefore, the methods of this paper have a distinctly algebraic character and are somewhat more elementary than was required in the cited references.

The starting point of this work is the result of Heilbronn and Linfoot [5] who proved that all imaginary quadratic fields of class number 1 have discriminants  $\geq -200$ , with one possible exception. It is an immediate consequence of this result that there are at least 9 and at most 10 imaginary quadratic fields of class number 1. It is interesting to note, however, that all known proofs that there are exactly 9 imaginary quadratic fields of class number 1 make no use of the result of Heilbronn and Linfoot, but rather classify the relevant fields directly.

The work of Heilbronn and Linfoot was generalized by T. Tatzuzaawa [9], who proved that for  $h \geq 1$ ,  $d =$  the discriminant of an imaginary quadratic field of class

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number  $\leq h$ , then

$$|d| \leq 2100h^2 \log^2(13h),$$

with one possible exceptional  $d$ .

Recently, J. Sunley [8] has generalized Tatzawa's result as follows:

**THEOREM [S].** *Let  $h \geq 1$  and let  $K$  be a fixed totally real algebraic number field. Further, let  $L$  be a totally imaginary quadratic extension of  $K$ , having class number  $h_L$  and discriminant  $d_L$ . Then there exists an effectively computable constant  $c = c(K, h)$  such that if  $h \leq h_L$ , then  $|d_L| \leq c$ , with the possible exception of one field  $L$ .*

In this paper, we will prove

**THEOREM 1.** *Let all notations be as in [S], and assume that  $K$  is normal. Then the exceptional field (if it exists) must be normal over  $\mathbb{Q}$ .*

**THEOREM 2.** *Let all notations be as in [S], and assume that  $K$  is normal. Assume that  $L/\mathbb{Q}$  is normal.*

(1) *If  $h_L = 1$ , then either  $L = K((-p)^{1/2})$ , where  $\mathbb{Q}((-p)^{1/2})$  is an imaginary quadratic field of class number 1, or  $L$  belongs to a finite, effectively determined collection of fields.*

(2) *Let  $K$  be a real quadratic field of class number 1 and fundamental unit of norm  $-1$ . If  $h_L = 2$ , then either  $L = K((-pq)^{1/2})$ , where  $\mathbb{Q}((-pq)^{1/2})$  is an imaginary quadratic field of class number 2, or  $L$  belongs to a finite, effectively determined collection of fields.*

As immediate consequences of [S], Theorems 1 and 2, and the effective classification of imaginary quadratic fields of class number 1 and 2, we get

**THEOREM 3.** *Assume that  $K$  is normal. There exists an effectively determined constant  $c = c(K, h)$  such that if  $L$  has class number  $h$  and  $L/\mathbb{Q}$  is nonnormal, then  $|d_L| \leq c$ .*

**THEOREM 4.** *Assume that  $K$  is normal. It is possible to effectively determine all those  $L$  for which  $h_L = 1$ .*

**THEOREM 5.** *Let  $K$  be a real quadratic field of class number 1 whose fundamental unit has norm  $-1$ . Then it is possible to effectively determine all those  $L$  for which  $h_L = 2$ .*

It may be possible to generalize Theorem 5 to the case of arbitrary totally real  $K$ , but we do not see how to accomplish this at the present time.

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**2. Proof of Theorem 1.** Throughout this paper, let  $K$  be a normal totally real algebraic number field of degree  $n$  and class number  $h_K$ . Let  $N = N_{K/\mathbb{Q}}$  and let

$x \rightarrow x^{(i)}$  ( $1 \leq i \leq n$ ) denote the distinct conjugation maps of  $K$ . Further, let  $L = K(\delta^{1/2})$  be a totally imaginary quadratic extension of  $K$  and let  $h_L$  denote the class number of  $L$ .

In order to prove Theorem 1, let us assume that  $L/\mathcal{Q}$  is not normal. Since  $K$  is normal, the conjugate fields of  $L$  are all of the form  $L_i = K((\delta^{(i)})^{1/2})$  ( $1 \leq i \leq n$ ), and since  $L$  is not normal, there exists an  $i$  ( $1 \leq i \leq n$ ) such that

$$K(\delta^{1/2}) \neq K((\delta^{(i)})^{1/2}).$$

Since  $L$  is isomorphic to  $L_i$  over  $\mathcal{Q}$ ,  $L$  and  $L_i$  have the same discriminant  $d$ . Thus, if  $|d| > c$ , where  $c$  is as in [S], then we get an immediate contradiction to [S]. Therefore,  $|d| \leq c$  and  $L$  cannot be the exception field. This completes the proof of Theorem 1.

### 3. Preliminaries to the Proof of Theorem 2.

LEMMA 3-1. *If  $L$  has class number 1, then  $K$  has class number 1. If  $L$  has class number 2, then  $K$  has class number 1 or 2.*

**Proof.** Let  $K^*$  (resp.  $L^*$ ) denote the Hilbert class field of  $K$  (resp.  $L$ ). Then  $K^*L$  is an unramified, abelian extension of  $L$ , so that  $K^*L \subseteq L^*$ . But since  $K^*/K$  is unramified,  $K^*$  is totally real and thus  $K^*$  is linearly disjoint from  $L$  over  $K$ . Therefore,  $\deg(K^*L/L) = \deg(K^*/K) = h_K$ . But if  $L$  has class number 1,  $L^* = L$ , so that  $\deg(K^*L/L) = 1 \Rightarrow h_K = 1$ . On the other hand, if  $L$  has class number 2, then  $\deg(L^*/L) = 2$  and therefore  $h_K = 1$  or 2.

For the sake of simplicity, let us assume throughout the remainder of this paper that if  $L$  has class number 2, then  $K$  has class number 1.

By Lemma 3-1 and the assumption, the ring of integers  $\mathfrak{D}_K$  of  $K$  is a unique factorization domain and therefore we may write  $L = K(\mu^{1/2})$  where  $\mu$  is a square-free integer of  $K$ . Let

$$2\mathfrak{D}_K = \mathfrak{p}_1^{\kappa_1} \cdots \mathfrak{p}_t^{\kappa_t}$$

where  $\mathfrak{p}_i$  is a  $K$ -prime. Suppose that

$$\mathfrak{p}_i \nmid \mu \mathfrak{D}_K \quad (1 \leq i \leq s), \quad \mathfrak{p}_i \mid \mu \mathfrak{D}_K \quad (s+1 \leq i \leq t).$$

Let  $r_i$  ( $1 \leq i \leq s$ ) denote the largest nonnegative integer  $\leq \kappa_i$  such that there exists  $u_i \in K$  such that  $\mu \equiv u_i^2 \pmod{\mathfrak{p}_i^{2r_i}}$ . Then, by a classical result of Kummer theory,

$$(1) \quad d_{L/K} = \prod_{i=1}^s \mathfrak{p}_i^{2(\kappa_i - r_i)} \cdot \prod_{i=s+1}^t \mathfrak{p}_i^{2\kappa_i} \cdot \mu \mathfrak{D}_K,$$

where  $d_{L/K}$  denotes the relative discriminant of the extension  $L/K$ . Since the  $\mathfrak{p}_i$  are principal, we can choose an integer  $\delta \in K$  such that

$$(2) \quad L = K(\delta^{1/2}), \quad d_{L/K} = \delta \mathfrak{D}_K.$$

Note that  $\delta$  is determined up to multiplication by the square of a unit of  $K$ .

Choose  $b \in \mathfrak{D}_K$  so that

$$b \equiv u_i \pmod{\mathfrak{p}_i^t} \quad (1 \leq i \leq s),$$

let  $\mathfrak{p}_i = \pi_i \mathfrak{D}_K$  ( $1 \leq i \leq t$ ), and let  $\lambda = \prod_{i=1}^s \pi_i^{r_i}$ . Then  $1, (b - \mu^{1/2})/\lambda$  is an integral basis of  $L$ . In particular, every integer of  $L$  can be written in the form

$$(3) \quad (x + y\mu^{1/2})/\lambda \quad (x, y \in \mathfrak{D}_K).$$

**LEMMA 3-2.** *Let  $U_K$  (resp.  $U_L$ ) denote the group of units of  $\mathfrak{D}_K$  (resp.  $\mathfrak{D}_L$ ). If  $|N(\mu)| > 4^n$ , then  $U_K = U_L$ .*

**Proof.** Let  $u = (x + y(\mu^{1/2}))/\lambda \in U_L$ . Then  $|N_{L/Q}(\lambda)| = 1$ , so that

$$(4) \quad |N(x^2 - \mu y^2)| = |N_{L/Q}(\lambda)| = |N(\lambda)|^2 \leq 4^n.$$

However, if  $y \neq 0$ ,

$$|N(x^2 - \mu y^2)| \geq |N(\mu y^2)| > 4^n |N(y)|^2 \geq 4^n,$$

which is a contradiction to (4). Therefore,  $y=0$  and  $u \in U_K$ . Thus,  $U_L \subseteq U_K$ . The converse inclusion is obvious.

**LEMMA 3-3.** *Let  $L^*$  = the Hilbert class field of  $L$  and assume that  $h_L=2$ . Then there exists an integer  $\alpha \in K$  such that*

$$(i) \quad L^* = K(\alpha^{1/2}, \delta^{1/2}),$$

$$(ii) \quad d_{K(\alpha^{1/2})/K} = \alpha \mathfrak{D}_K.$$

**Proof.** Since  $\deg(L^*/L)=2$ , there exist  $\alpha, \beta \in K$  such that

$$L^* = L((\alpha + \beta \delta^{1/2})^{1/2}).$$

If  $\beta \neq 0$ , then  $L^{**} = L((\alpha + \beta \delta^{1/2})^{1/2}, (\alpha - \beta \delta^{1/2})^{1/2})$  is an unramified extension of  $L$  of degree 4 (since  $L((\alpha + \beta \delta^{1/2})^{1/2}) = L((\alpha - \beta \delta^{1/2})^{1/2})$  implies that  $(\alpha + \beta \delta^{1/2})^{1/2} = a(\alpha - \beta \delta^{1/2})^{1/2} + b$  ( $a, b \in L$ )) which is impossible since  $L$  has class number 2. Therefore,  $\beta=0$ . Using the same reasoning as led to (1), we may choose  $\alpha$  to be an integer in  $K$  and alter it by the square of  $K$  to ensure that (ii) holds.

The next result requires Furuta's formula [2] for the genus number of an abelian extension; so let us briefly review Furuta's theory. Let  $M$  and  $N$  be number fields,  $M/N$  abelian. Let  $M^*$  = the maximal unramified extension of  $M$  which is normal over  $N$ . Then  $M^*$  is called the *genus field* of the extension  $M/N$ , and the degree of  $M^*$  over  $M$  is called the *genus number* of the extension  $M/N$ . Furuta has found the following beautiful formula for the genus number:

$$(5) \quad \deg(M^*/M) = \frac{h_N \prod_{\mathfrak{p}} e'_{\mathfrak{p}}}{\deg(M_0/N)[U_N:U_M^0]},$$

where

$h_N$  = the class number of  $N$ ,

$\mathfrak{p}$  runs over all  $N$ -primes,

$M_0$  = the maximum abelian extension of  $N$  contained in  $M$ ,

$e'_p$  = the ramification index of the maximum abelian extension of  $N_p$  contained in  $M_p$ , where  $p$  is any  $M$ -prime dividing  $p$ , and  $N_p$  and  $M_p$  are the respective completions of  $N$  and  $M$  at  $p$  and  $p$ ,

$U_N$  = the group of units of  $\mathfrak{D}_N$ ,

$U_M^0$  = the group of units of  $\mathfrak{D}_N$  which are locally everywhere norms of units in  $M$ .

PROPOSITION 3-4. Assume that  $|N(\mu)| > 4^n$ .

(a) If  $h_L = 1$ , then  $d_{L/K}$  is divisible by exactly one prime.

(b) If  $h_L = 2$ , then  $d_{L/K}$  is divisible by exactly two primes.

**Proof.** As above, let  $L^*$  denote the Hilbert class field of  $L$ . In case (a),  $L^* = L$ , so that  $L^*$  is abelian over  $K$ . In case (b), Lemma 3-3 asserts that  $L^*$  is abelian over  $K$ . In either case, the genus field of the extension  $L/K$  is  $L^*$  and the genus number is  $h_L$ . Let us use (5) to compute the genus number in a different way. We set  $M = L$ ,  $N = K$ ,  $M_0 = L$ . Suppose that  $d_{L/K}$  is divisible by  $r$  distinct finite  $K$ -primes. Since  $K$  is totally real and  $L$  is totally imaginary, every infinite  $K$ -prime ramifies in  $L$ . Therefore,

$$\prod_p e'_p = 2^{r+n}.$$

Moreover, by Lemma 3-1, and our restrictive assumption in case  $h_L = 2$ , we have  $h_K = 1$ . Further, by Hasse's theorem, an element of  $K^\times$  which is everywhere a local norm from  $L$  is a global norm from  $L$ . Therefore, by Lemma 3-2,

$$U_L^0 = N_{L/K} U_K \supseteq U_K^2.$$

But by Dirichlet's theorem,  $U_K \approx \{\pm 1\} \times \mathbb{Z}^{n-1}$ , so that  $[U_K : U_L^0] \geq 2^n$ . Assembling all the data into Furuta's formula, we get  $h_L \geq 2^{r-1}$ , from which (a) and (b) follow immediately, since  $h_L = 2$  implies that  $K$  is a real quadratic field with fundamental unit of norm  $-1$ .

In the remainder of this section, we will study the case  $h_L = 2$  more thoroughly. We found that in this case  $L^* = L(\alpha^{1/2}, \delta^{1/2})$ . Let  $K' = K(\alpha^{1/2})$ . Then by the transitivity formula for the discriminant, we see that

$$d_{L^*/K} = d_{L/K}^2 N_{L/K} d_{L^*/L} = \delta^2 \mathfrak{D}_K$$

since  $L^*/L$  is unramified. On the other hand,

$$d_{L^*/K} = d_{K'/K}^2 \cdot N_{K'/K} d_{L^*/K'} = \alpha^2 N_{K'/K} d_{L^*/K'}.$$

Therefore,  $\alpha | \delta$ . Let  $\beta = \delta/\alpha$ . Then  $\beta \in \mathfrak{D}_K$  and  $L^* = K(\alpha^{1/2}, \beta^{1/2})$ ,  $\alpha\beta = \delta$ ,  $d_{L/K} = \delta \mathfrak{D}_K$ . Set  $L' = K(\alpha^{1/2})$  and  $L'' = K(\beta^{1/2})$ . We have already normalized  $\alpha$  so that  $d_{L'/K} = \alpha \mathfrak{D}_K$ . We claim that  $\alpha$  and  $\beta$  are each divisible by one prime. This will follow from

THEOREM 3-5. One of the fields  $L', L''$  is totally real and the other is totally imaginary.

**Proof.** Let  $J_K$  denote the idele group of  $K$ . If  $F/K$  is an abelian extension, let  $H(F)$  denote the admissible subgroup of  $J_K$  corresponding to  $F$ . Let  $\mathfrak{p}_{\infty,0}, \dots, \mathfrak{p}_{\infty,r}$  denote the infinite primes of  $K$ , and let

$$\delta_{\infty,i} = (\alpha_{\mathfrak{p},i}) \in J_K \quad (0 \leq i \leq r)$$

be the idele defined by

$$\begin{aligned} \alpha_{\mathfrak{p},i} &= 1 & (\mathfrak{p} \neq \mathfrak{p}_{\infty,i}) \\ &= -1 & (\mathfrak{p} = \mathfrak{p}_{\infty,i}). \end{aligned}$$

Since  $L$  is a quadratic extension of  $K$ ,  $[J_K : H(L)] = 2$ . Moreover, since  $K$  is totally real and  $L$  is totally imaginary,  $\delta_{\infty,0} \notin H(L)$ . Therefore, we have the coset decomposition

$$(6) \quad J_K = H(L) \cup \delta_{\infty,0}H(L).$$

Let  $S$  denote the set of all  $K$ -primes  $\mathfrak{p}$  such that  $f_{\mathfrak{p}}(L/K) = 1$ ,  $f_{\mathfrak{p}}(L^*/K) = 2$ , where  $f_{\mathfrak{p}}$  denotes the residue class degree. Since  $L^* = K(\alpha^{1/2}, \beta^{1/2})$ ,  $L^*/K$  is an abelian extension with the Klein 4-group as Galois group. Therefore, by Tchebotarev's density theorem,  $S$  has Dirichlet density  $\frac{1}{4}$  and, in particular, is infinite. Therefore, let us choose  $\mathfrak{p} \in S$  such that (i)  $\mathfrak{p}$  is finite and (ii)  $\mathfrak{p}$  does not ramify in  $L^*$ . Let  $\pi$  be a local uniformizing parameter at  $\mathfrak{p}$ . Then  $\pi$  is a local norm from  $L_{\mathfrak{p}}$ , but is not a local norm from  $L^*_{\mathfrak{p}}$ . Therefore, the idele

$$\begin{aligned} b &= (1, \dots, 1, \pi, 1, \dots, 1) \in J_K \\ &\quad \uparrow \\ &\quad \mathfrak{p} \end{aligned}$$

is in  $H(L) - H(L^*)$ . However, by class field theory,  $H(L^*) \subseteq H(L)$  and  $[H(L) : H(L^*)] = 2$ . Therefore,

$$H(L) = H(L^*) \cup bH(L^*).$$

Thus, by (6), we see that

$$(7) \quad J_K = H(L^*) \cup bH(L^*) \cup \delta_{\infty,0}H(L^*) \cup \delta_{\infty,0}bH(L^*).$$

Furthermore, since  $L^*/K$  is abelian and  $L^*$  is the Hilbert class field of  $L$ ,  $L^*$  is the maximal unramified extension of  $L$  which is abelian over  $K$ . Thus,  $L^*$  is the genus field of  $L$  over  $K$ . Therefore, by [2, Proposition 1], we know that

$$(8) \quad H(L^*) = K^* N_{L/K} U_L,$$

where  $N_{L/K}$  denote the idele norm from  $L$  to  $K$  and  $U_L$  is the group of unit ideles of  $L$ . By class field theory, we have

$$(9) \quad H(L') = H(L^*) \cup \delta_{\infty,0}H(L^*),$$

$$(10) \quad H(L'') = H(L^*) \cup \delta_{\infty,0}bH(L^*).$$

Let us show that  $L'$  is totally real and that  $L''$  is totally imaginary. It clearly suffices to prove the former since  $\alpha\beta$  is totally negative. But

$$(11) \quad L' \text{ is totally real} \Leftrightarrow \delta_{\infty,i} \in H(L') \quad (1 \leq i \leq r).$$

Let us assume that  $L'$  is not totally real. Then for some  $i$  ( $1 \leq i \leq r$ ), we have  $\delta_{\infty,i} \notin H(L')$ . By (7) and (9), we must have either (a)  $\delta_{\infty,i} \in bH(L^*)$  or (b)  $\delta_{\infty,i} \in \delta_{\infty,0}bH(L^*)$ . On account of (8), there exist  $k \in K^*$ ,  $u \in N_{L/K}U_L$ , such that either

$$(a) \quad \delta_{\infty,i}bu = k \text{ or}$$

$$(b) \quad \delta_{\infty,i}\delta_{\infty,0}bu = k.$$

Since  $L^*/L$  is unramified,  $N_{L/K}U_L = N_{L^*/K}U_{L^*} \subseteq N_{L'/K}U_{L'}$ . Therefore,  $u \in N_{L'/K}U_{L'}$ . Let  $[\cdot, L'/K]: J_K \rightarrow \{\pm 1\}$  denote the global norm residue symbol for  $L'/K$ . Since  $H(L')$  is the kernel of  $[\cdot, L'/K]$  and since  $\delta_{\infty,0} \in H(L')$ ,  $u \in H(L')$ ,  $k \in H(L')$ , we see that  $[b, L'/K] = -1 \Rightarrow \mathfrak{p}$  is inert in  $L'/K \Rightarrow \mathfrak{p}$  is decomposed in  $L''/K$  (since  $f_{\mathfrak{p}}(L/K) = 1$ ,  $f_{\mathfrak{p}}(L^*/K) = 2$ ). Thus, with finitely many exceptions, every prime of  $S$  is decomposed in  $L''/K$ . Therefore, with finitely many exceptions, every  $K$ -prime  $\mathfrak{p}$  which decomposes in  $L$  decomposes in  $L'$ . Therefore, by Bauer's theorem<sup>(2)</sup>  $L \subseteq L'' \Rightarrow L = L''$ , which gives a contradiction, since then  $L''$  is totally imaginary and  $L'$  is totally real.

Let us assume that  $\alpha$  is totally positive and  $\beta$  is totally negative. Recall that we normalized  $\alpha$  so that  $d_{L'/K} = \alpha \mathfrak{D}_K$ . Claim that

$$(12) \quad d_{L^*/K} = \beta \mathfrak{D}_K.$$

We have already shown that

$$(13) \quad d_{L'L^*/K} = d_{L^*/K} = \delta^2 \mathfrak{D}_K.$$

But by Theorem 3-5, the fields  $L'$  and  $L''$  are linearly disjoint over  $K$ . Therefore,

$$\mathfrak{D}_{L'L^*/K} = \mathfrak{D}_{L'/K} \cdot \mathfrak{D}_{L^*/K},$$

where  $\mathfrak{D}_{M/N}$  denotes the difference of the extension  $M/N$ . Thus,  $d_{L'L^*/K} = d_{L'/K}^2 \cdot d_{L^*/K}^2$ , and therefore  $d_{L^*/K} = \beta \mathfrak{D}_K$ .

**COROLLARY 3-6.** Assume that  $h_L = 2$  and  $|N(\mu)| > 4^n$ . Then  $\alpha$  and  $\beta$  are each divisible by exactly one  $K$ -prime.

**Proof.** Since  $K$  has class number 1,  $\alpha$  and  $\beta$  are both divisible by at least one  $K$ -prime, since otherwise  $L'$  or  $L''$  would be an unramified abelian extension of  $K$ . However,  $\alpha\beta = \delta$  and, by Proposition 3-4(b),  $\delta$  is divisible by exactly 2  $K$ -primes. Therefore, it suffices to show that  $\alpha$  and  $\beta$  cannot both be divisible by the same  $K$ -prime  $\mathfrak{p}$ . But if  $\mathfrak{p}|\alpha$  and  $\mathfrak{p}|\beta$ , then  $\mathfrak{p}$  is totally ramified in  $L'L'' = L^*$ . But since  $L^*/L$

<sup>(2)</sup> See M. Bauer, *Zur Theorie der Algebraischen Zahlkörper*, Math. Ann. 77 (1916), or H. Hasse, *Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper*, II, Jber. Deutsch. Math. Verein. 35 (1926), §25.

is unramified, no  $K$ -prime can be totally ramified in  $L^*$ . Thus, a contradiction is reached and the corollary is established.

**4. Proof of Theorem 2 in Case  $h_L = 1$ .** We assume throughout this section that  $L = K(\delta^{1/2})$  is of class number 1 and normal over  $\mathcal{Q}$ . If  $x \in K^*$ , we will denote by  $(x)$  the principal  $K$ -ideal generated by  $x$ . Let  $\varepsilon_1, \dots, \varepsilon_r$  ( $r = n - 1$ ) be a basis for the free part of the group of  $K$ -units.

*Reduction 1.* We may assume that  $|N(\mu)| > 4^n$ .

For there are only finitely many integral  $K$ -ideals  $\mathfrak{A}$  such that  $N\mathfrak{A} \leq 4^n$ . Since  $K$  has class number 1, every such ideal is principal. Let  $\eta$  run through some fixed set of generators for these ideals. Then, if  $|N(\delta)| \leq 4^n$ ,  $L$  is one of the finite collection of fields  $K((\varepsilon\eta)^{1/2})$ , where

$$(14) \quad \varepsilon = \prod_{j=1}^t \varepsilon_{i_j} \quad (1 \leq i_1 < i_2 < \dots < i_t \leq r).$$

These fields may be individually tested to determine which has class number 1 (using the Minkowski bound, say). Thus, we may restrict ourselves to  $L$  such that  $|N(\delta)| > 4^n$ .

By Proposition 3-4(a) and Reduction 1, we know that  $\delta$  is divisible by exactly one  $K$ -prime  $\mathfrak{p}$ .

*Reduction 2.* We may assume that  $\mathfrak{p}$  is not a divisor of 2.

For if  $\mathfrak{p}_1, \dots, \mathfrak{p}_s$  are the  $K$ -primes dividing 2 and if  $\pi_i$  is a generator of  $\mathfrak{p}_i$ , then the fields  $L$  for which  $\delta$  is divisible by some  $\mathfrak{p}_i$  are of the form  $K((\varepsilon\pi_i)^{1/2})$  or  $K(\varepsilon^{1/2})$ , where  $\varepsilon$  is given by (14). Once again these fields may be individually studied.

*Reduction 3.* We may assume that  $\mathfrak{p} \nmid d_K$ .

For if  $\mathfrak{p} \mid d_K$  and if the primes dividing  $d_K$  are  $q_1, \dots, q_n$ , then  $L$  is of the form  $K((\varepsilon\lambda_i)^{1/2})$ , where  $\lambda_i$  is a generator of  $q_i$  and  $\varepsilon$  is of the form (14).

Henceforth, we will assume that Reductions 1-3 have been carried out. By Reduction 2 and equation (1),  $\delta = \mu$  is irreducible. Let  $p$  be the rational prime which  $\mathfrak{p}$  divides. Let us renumber the conjugates  $\delta^{(i)}$  so that the distinct  $K$ -primes dividing  $p$  are  $(\delta^{(1)}), \dots, (\delta^{(r)})$ . Then, by Reduction 3,  $(p) = (\delta^{(1)}) \cdots (\delta^{(r)})$ . Thus,  $\delta^{(1)} \cdots \delta^{(r)} = \zeta p$  for some  $K$ -unit  $\zeta$ . If  $r$  is even, then since  $\delta$  is totally negative, we have  $\zeta p$  is totally positive. Since  $L$  is normal over  $\mathcal{Q}$ ,  $(\delta^{(i)})^{1/2} \in L$  ( $1 \leq i \leq r$ )  $\Rightarrow (\zeta p)^{1/2} \in L \Rightarrow (\zeta p)^{1/2} \in K$  since  $\zeta p$  is totally positive. But then,  $p$  ramifies in  $K$ , which contradicts Reduction 3. Thus  $r$  is odd and  $\zeta$  is totally negative. Assume that  $\zeta \neq -x^2$  for some  $x \in K$ . Then, if  $p \equiv 1 \pmod{4}$ ,  $L = K((\zeta p)^{1/2})$  and  $L(p^{1/2}) = K(\zeta^{1/2}, p^{1/2})$  is an unramified extension of  $L$  of degree 2. But this contradicts the fact that  $L$  has class number 1. On the other hand, if  $p \equiv 3 \pmod{4}$ , then  $L = K((\zeta p)^{1/2})$  and  $L((-p)^{1/2}) = K((- \zeta)^{1/2}, (-p)^{1/2})$  is an unramified quadratic extension of  $L$ , which is again a contradiction to the fact that  $L$  has class number 1. Thus,  $\zeta = -x^2$  for some  $x \in K$  and  $(-p)^{1/2} \in L \Rightarrow L = K((-p)^{1/2})$ .

*Claim.*  $\mathcal{Q}((-p)^{1/2})$  has class number 1.



Let  $H^*$  denote the Hilbert class field of  $\mathcal{Q}((-p)^{1/2})$ . Then  $K \cdot H^*$  is an unramified, abelian extension of  $K((-p)^{1/2})$ . However, since  $K((-p)^{1/2}) = L$  has class number 1, we see that

$$(15) \quad K \cdot H^* = K((-p)^{1/2}) \Rightarrow H^* \subseteq K((-p)^{1/2}).$$

Moreover, it is clear that

$$(16) \quad (-p)^{1/2} \in H^*.$$

Let  $H_0^*$  denote the maximal real subfield of  $H^*$ . Then  $H^* = H_0^* \mathcal{Q}((-p)^{1/2})$ ,  $H_0^* \subseteq K$ , by (15) and (16). Assume that  $\deg(H_0^*/\mathcal{Q}) > 1$ . Then there exists a  $\mathcal{Q}$ -prime  $q \neq p$  such that  $q$  ramifies in  $H_0^*$ . For by Minkowski's theorem, there exists a  $\mathcal{Q}$ -prime  $q$  which ramifies in  $H_0^*$  and  $q \neq p$  by Reduction 3. Moreover, if 2 does not ramify in  $\mathcal{Q}((-p)^{1/2})/\mathcal{Q}$ , we see that  $q$  ramifies in  $H_0^*$ , but does not ramify in  $\mathcal{Q}((-p)^{1/2})/\mathcal{Q}$ . But this is a contradiction to the fact that  $H^*/\mathcal{Q}((-p)^{1/2})$  is unramified. Thus,  $\deg(H_0^*/\mathcal{Q}) = 1 \Rightarrow H_0^* = \mathcal{Q} \Rightarrow H^* = \mathcal{Q}((-p)^{1/2}) \Rightarrow \mathcal{Q}((-p)^{1/2})$  has class number 1. If 2 ramifies in  $\mathcal{Q}((-p)^{1/2})/\mathcal{Q}$ , then  $p \equiv 1 \pmod{4}$  and  $p^{1/2} \in H_0^*$ . Thus  $p$  ramifies in  $K$ , contrary to Reduction 3.

From the claim and Stark's theorem, we know that the only possibilities for  $p$  are  $p=3, 7, 11, 19, 43, 67, 163$ . Thus, except for the fields set aside in making Reductions 1-3, the only possibilities for  $L$  are

$$L = K((-3)^{1/2}), K((-7)^{1/2}), K((-11)^{1/2}), K((-19)^{1/2}), K((-43)^{1/2}), \\ K((-67)^{1/2}), K((-163)^{1/2}).$$

This completes the proof of Theorem 2 in case  $h_L = 1$ .

**5. Proof of Theorem 2 in Case  $h_L = 2$ .** Throughout this section, we will assume that  $K$  is a real quadratic field of discriminant  $d$ , whose fundamental unit  $\varepsilon$  has norm  $-1$ . Further, we will assume that  $L$  is a totally imaginary quadratic extension of  $K$  of class number 2 such that  $L/\mathcal{Q}$  is normal.

From the results of §3, we know that  $L^* = K((\alpha^{1/2}, \beta^{1/2}))$ , where  $\alpha, \beta \in K$ , and  $\alpha$  is totally negative and  $\beta$  totally positive. Moreover, since  $L/\mathcal{Q}$  is normal and  $L$  has class number 2,  $L^*$  is normal over  $\mathcal{Q}$ . As in §4, it suffices to consider the case where  $|N(\delta)| > 4^n$ . We will assume throughout that we are in this case. Then, from the results of §3, we know that  $\alpha$  and  $\beta$  are both divisible by exactly one  $K$ -prime. Suppose that  $\alpha$  is divisible by  $\mathfrak{p}$  and  $\beta$  by  $\mathfrak{q}$ . Let  $p$  and  $q$  be, respectively, the  $\mathcal{Q}$ -primes dividing  $\mathfrak{p}$  and  $\mathfrak{q}$ . Also note that, since  $K$  has class number 1,  $d=8$  or  $d$  is an odd prime discriminant (that is,  $d=8$  or  $d$  is a prime and  $d \equiv 1 \pmod{4}$ ). Note that the situation we are considering actually occurs. In fact, there are 11 values of  $d$  less than 100 for which our assumptions are satisfied:

$$d = 5, 8, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97.$$

Let us consider four separate cases corresponding to  $p$  and  $q$  both odd;  $p=2$  and  $q$  odd;  $p$  odd and  $q=2$ ;  $p=q=2$ . First, however, let us collect some observations which are common to the four cases.

Let  $L_0^* = \mathcal{Q}(d^{1/2}, \beta^{1/2})$  denote the maximal real subfield of  $L^*$ .

A. *There exist no fields  $L$  for which  $p$  = the unique prime dividing  $d$ .*

For suppose that  $p$  = the unique prime dividing  $d$ . If  $d$  is odd, then  $p$  is odd and  $\alpha$  is an irreducible element of  $\mathfrak{D}_K$ . Therefore,  $\alpha = -\varepsilon^a d^{1/2}$  for some  $a \in \mathbb{Z}$ . But since  $\alpha$  is totally negative, and  $N(\varepsilon) = -1$ , we see that  $a$  is even and, without loss of generality, we may assume that  $\alpha = (-\sqrt{d})^{1/2}$ . Since  $L^*/\mathcal{Q}$  is normal,  $d^{1/4}, (-1)^{1/2} \in L^* \Rightarrow L^* = \mathcal{Q}(d^{1/4}, (-1)^{1/2})$  since  $\deg(L^*/\mathcal{Q}) = 8$ . Therefore,

$$L = K((-4\sqrt{d})^{1/2}).$$

But for this field,  $p=2$ . Thus, we contradict the fact that  $p$  is odd. If  $d=8$ , then  $p=2$  and  $\alpha = \pm \varepsilon^a (2^{1/2})^b$  ( $a, b \in \mathbb{Z}$ ). Since  $\alpha$  is totally negative and  $N(\varepsilon) = -1$ , we may assume that  $\alpha = -(2^{1/2})^b$ . Since  $L^*/\mathcal{Q}$  is normal,  $((\sqrt{2})^b)^{1/2} \in L^*$  and thus  $L^* = \mathcal{Q}(2^{1/2}, (-1)^{1/2}, ((\sqrt{2})^b)^{1/2}, \beta^{1/2})$ . If  $b$  is odd, then

$$L^* = \mathcal{Q}(2^{1/4}, (-1)^{1/2}, \beta^{1/2}) = \mathcal{Q}(2^{1/4}, (-1)^{1/2})$$

since  $\deg(L^*, \mathcal{Q}) = 8$ . However, only one  $K$ -prime then ramifies in  $L^*$ , which contradicts Proposition 3-4(b). Thus  $b$  is even and  $L^* = \mathcal{Q}(2^{1/2}, (-1)^{1/2}, \beta^{1/2})$ . However, since  $\deg(L^*/\mathcal{Q}) = 8$ ,  $\beta \in \mathcal{Q}$  and thus since  $\beta$  is totally positive,

$$L^* = \mathcal{Q}(2^{1/2}, (-1)^{1/2}, q^{1/2}).$$

Therefore,  $L = K((-q)^{1/2}) = K((-8q)^{1/2})$ .

B.  $p=q=2$  occurs for only a finite number of fields and these can be effectively determined.

This is clear.

C. *There are no fields  $L$  for which  $p=q$  and both are odd.*

For if  $p=q$ , then either  $\beta = -\varepsilon^a \alpha$  or  $-\varepsilon^a \alpha'$ , where  $\alpha'$  denotes the conjugate of  $\alpha$  over  $K$ . Since  $\beta$  is totally positive, and  $N(\varepsilon) = -1$ , we see that  $a$  is even. And since  $L^*/\mathcal{Q}$  is normal,  $(\pm \alpha)^{1/2}, (\pm \alpha')^{1/2} \in L^* \Rightarrow (-1)^{1/2} \in L^* \Rightarrow 2$  ramifies in  $L^*$ . This contradicts the fact that  $d, p, q$  are all odd.

D. *There exists at most one field  $L$  for which  $q=d$ , namely*

$$L^* = \mathcal{Q}(d^{1/4}, (-1)^{1/2}), \quad L = K((-4\sqrt{d})^{1/2}).$$

This follows from the proof of A.

Case I.  $p, q$  odd. In this case, both  $\alpha$  and  $\beta$  are irreducible. Upon examining the factorization of  $p$  in  $K$ , we see that  $(p) = (\alpha)^{e_1} (\alpha')^{e_2}$ , for some integers  $e_i = 0, 1, 2$ , where  $(x)$  denotes the  $K$ -ideal generated by  $x$ . Since  $L^*/\mathcal{Q}$  is normal,  $(\alpha')^{1/2} \in L^* \Rightarrow$  there exists a  $K$ -unit  $\zeta$  of constant signature such that  $(\zeta p)^{1/2} \in L^*$ . If  $\zeta$  is totally positive,  $p$  ramifies in  $L_0^*$ , so that  $p=d$  or  $p=q$ . This is a contradiction by A and C. Thus  $\zeta$  is totally negative and of the form  $-\varepsilon^a$ ,  $a$  even. Thus,  $(-p)^{1/2} \in L^*$ . Applying to  $q$  the same reasoning as we applied to  $p$ , we see that  $(\eta q)^{1/2} \in L^*$  for some  $K$ -unit  $\eta$  of constant signature. If  $\eta$  is totally negative, then

$$\begin{aligned} (-\eta pq)^{1/2} \in L^* &\Rightarrow p \text{ ramifies in } L_0^* \quad (\text{since } -\eta pq \text{ is totally positive}) \\ &\Rightarrow p = d \quad \text{or} \quad p = q, \end{aligned}$$

which is a contradiction by A and C. Thus,  $\eta$  is totally positive. But then since  $N(\epsilon) = -1$ , we have  $\eta = \epsilon^a$  for  $a$  even which implies  $q^{1/2} \in L^*$ . Thus, we have shown that

$$Q(d^{1/2}, (-p)^{1/2}, q^{1/2}) \subseteq L^*, \quad Q(d^{1/2}, (-p)^{1/2}, q^{1/2}) = L^*,$$

since  $\deg(L^*/Q) = 8$  and  $d, p, q$  are distinct by A-D. Finally, we have  $L = K((-pq)^{1/2})$ . Moreover, since  $p$  and  $q$  are the only  $K$ -primes which ramify in  $L$ , we see that  $p$  and  $q$  are inert in  $Q(d^{1/2})$  and  $-p$  and  $q$  are prime discriminants.

*Case II.  $p$  odd,  $q=2$ .* As in Case I, we can prove that  $(-p)^{1/2} \in L^*$ . The difficulty in this case is that  $\beta$  may no longer be irreducible. Let  $\pi$  be a  $K$ -integer such that  $q = (\pi)$ . Then  $\beta = \epsilon^a \pi^b$  for some  $a, b \in \mathbb{Z}$ . If  $b$  is even, then  $a$  is even since  $\beta$  is totally positive and  $N(\epsilon) = -1$ . But then  $\beta^{1/2} \in K$ , which is a contradiction to the fact that  $\deg(L^*/K) = 4$ ,  $L^* = K(\alpha^{1/2}, \beta^{1/2})$ . Thus,  $b$  is odd and  $(\epsilon^a \pi)^{1/2} \in L^*$ . If 2 is inert in  $K$ , we may choose  $\pi = 2$ . Then, since  $N(\epsilon) = -1$  and  $\epsilon^a \pi$  is totally positive, we see that  $a$  is even and  $2^{1/2} \in L^*$ . If 2 decomposes in  $K$ ,  $(\epsilon^a \pi) \cdot (\epsilon^a \pi)' = 2$ , and thus since  $L^*/Q$  is normal,  $2^{1/2} \in L^*$ . Finally, reasoning as in Case I, we see that

$$L^* = Q(d^{1/2}, (-p)^{1/2}, 8^{1/2}), \quad L = K((-8p)^{1/2}),$$

where  $-p$  is an odd prime discriminant.

*Case III.  $p=2$ ,  $q$  odd.* As in Case I,  $q^{1/2} \in L^*$ . Let  $(\pi) = p$ ,  $\alpha = -\epsilon^a \pi^b$  where  $a, b \in \mathbb{Z}$ . If  $b$  is even, then  $a$  is even since  $N(\epsilon) = -1$  and  $(-1)^{1/2} \in L^*$ . On the other hand, if  $b$  is odd, then by reasoning as in Case II, we see that  $(-2)^{1/2} \in L^*$ . Thus, by reasoning as in Case I, we get

$$L^* = Q(d^{1/2}, (-4)^{1/2}, q^{1/2}) \quad \text{or} \quad L^* = Q(d^{1/2}, (-8)^{1/2}, q^{1/2})$$

and

$$L = K((-4q)^{1/2}) \quad \text{or} \quad L = K((-8q)^{1/2}),$$

where  $q$  is an odd prime discriminant.

*Case IV.  $p=q=2$ .* By B, we may neglect this case from consideration.

We will now prove

**THEOREM 5-1.** *Let  $d_1$  and  $d_2$  be distinct prime discriminants, neither equal to  $d$ , such that  $d_1 d_2 < 0$  and  $K((d_1 d_2)^{1/2})$  has class number 2. Then  $Q((d_1 d_2)^{1/2})$  has class number 2.*

**Proof.** Let  $H$  = the Hilbert class field of  $Q((d_1 d_2)^{1/2})$ ,  $H_0$  = the maximal real sub-field of  $H$ . The only primes which can ramify in  $H_0$  are the primes dividing  $d_1$  and  $d_2$ . Since  $K((d_1 d_2)^{1/2})$  has class number 2, its Hilbert class field is  $K((d_1)^{1/2}, (d_2)^{1/2})$ . Since  $H \cdot K$  is an unramified abelian extension of  $K((d_1 d_2)^{1/2})$ , we see that  $H \cdot K \subseteq K((d_1)^{1/2}, (d_2)^{1/2})$ . Let  $d_1 < 0$ ,  $d_2 > 0$ . Then  $(d_1)^{1/2} \in H$  and  $H = H_0 \cdot Q((d_1)^{1/2})$ ,  $H_0 \subseteq K((d_2)^{1/2})$ . Moreover,  $d$  ramifies in  $K$ , but since  $d \neq d_i$  ( $i=1, 2$ ),  $d$  does not

ramify in  $H_0$ . Therefore,  $H_0 \neq K((d_2)^{1/2})$ , so that

$$\begin{aligned} H_0 = \mathcal{Q}((d_2)^{1/2}) &\Rightarrow H = \mathcal{Q}((d_1)^{1/2}, (d_2)^{1/2}) \\ &\Rightarrow \deg(H/\mathcal{Q}((d_1 d_2)^{1/2})) = 2 \\ &\Rightarrow \mathcal{Q}((d_1 d_2)^{1/2}) \text{ has class number } 2. \end{aligned}$$

By Cases I–IV and Theorem 5-1, it suffices to effectively determine all imaginary quadratic fields of class number 2. But this is possible by the theorem of Baker and Stark. This completes the proof of Theorem 2 in case  $h_L = 2$ .

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